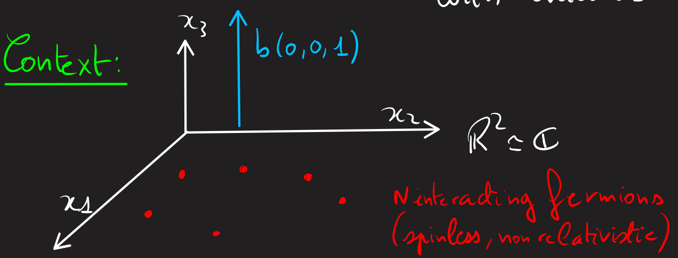
# Semi-classical limit of the 2D Hartree equation in a large magnetic field

with Nicolas Rougerie

## Context:




question: dynamic when $N, b \to +\infty$?

Goal: Hartree $\to$ Gyrokinetic transport

motivation: QHE

## Model:

magnetic Laplacian: $\mathcal{L}_b := (i\hbar\nabla + bA)^2 = \sum_{n\in\mathbb{N}} 2\hbar b\left(n+\frac{1}{2}\right)\Pi_n$ — $n^{th}$ Landau level projection

Symmetric gauge: $A := \frac{X^\perp}{2} = \frac{1}{2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}$, $\nabla\wedge A = (0,0,1)$, magnetic lenght: $\ell_b := \sqrt{\frac{\hbar}{b}}$

position operator in $\mathbb{R}^2$

Hartree equation: $i\hbar\partial_t\gamma = [\mathcal{L}_b + V + \omega * \rho_\gamma, \gamma]$ where $\gamma\in\mathfrak{L}^1(L^2(\mathbb{R}^2))$, $\mathrm{Tr}[\gamma]=1$, $\gamma\geq 0$ (density matrix)

and $V:\mathbb{R}^2\to\mathbb{R}$, $\omega:\mathbb{R}^2\to\mathbb{R}$; $\rho_\gamma(z) := \gamma(z,z)$

scaling: $\hbar\to 0$, $b\to+\infty$ s.t. $\hbar b\to 1$

Newton: $z'' = F + bz'^\perp \Rightarrow z(t) = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix} + \frac{F^\perp}{b}t$ — motion on time scale $\mathcal{O}(b)$

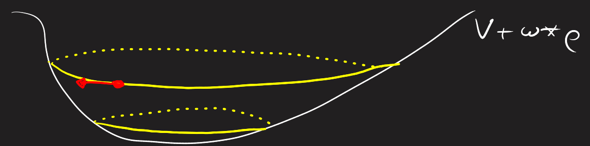
$\gamma_b(t) := \gamma(bt)$ then $\partial_t\gamma_b = \frac{1}{i\ell_b^2}[\mathcal{L}_b + V + \omega*\rho_{\gamma_b}, \gamma_b]$ (H)

Pauli principle: $\gamma_b \leq 2\pi\ell_b^2$, surfacic degeneracy of a LL: $1/2\pi\ell_b^2$

ex: $\gamma_b := \frac{1}{N}\sum_{i=1}^N |u_i\rangle\langle u_i|$ with $(u_i)_i \perp$, $N = \frac{1}{2\pi\ell_b^2}$ so $\mathrm{Tr}[\gamma_b] = 1$, $0\leq\gamma_b\leq\frac{1}{N} = 2\pi\ell_b^2$, say $\gamma_b$ is a FDM

$\mathcal{O}(1)$ volume

Gyrokinetic transport equation: $\partial_t\rho + \nabla^\perp(V+\omega*\rho)\cdot\nabla\rho = 0$
with $\rho:\mathbb{R}_+\times\mathbb{R}^2\to\mathbb{R}_+$




## Results:

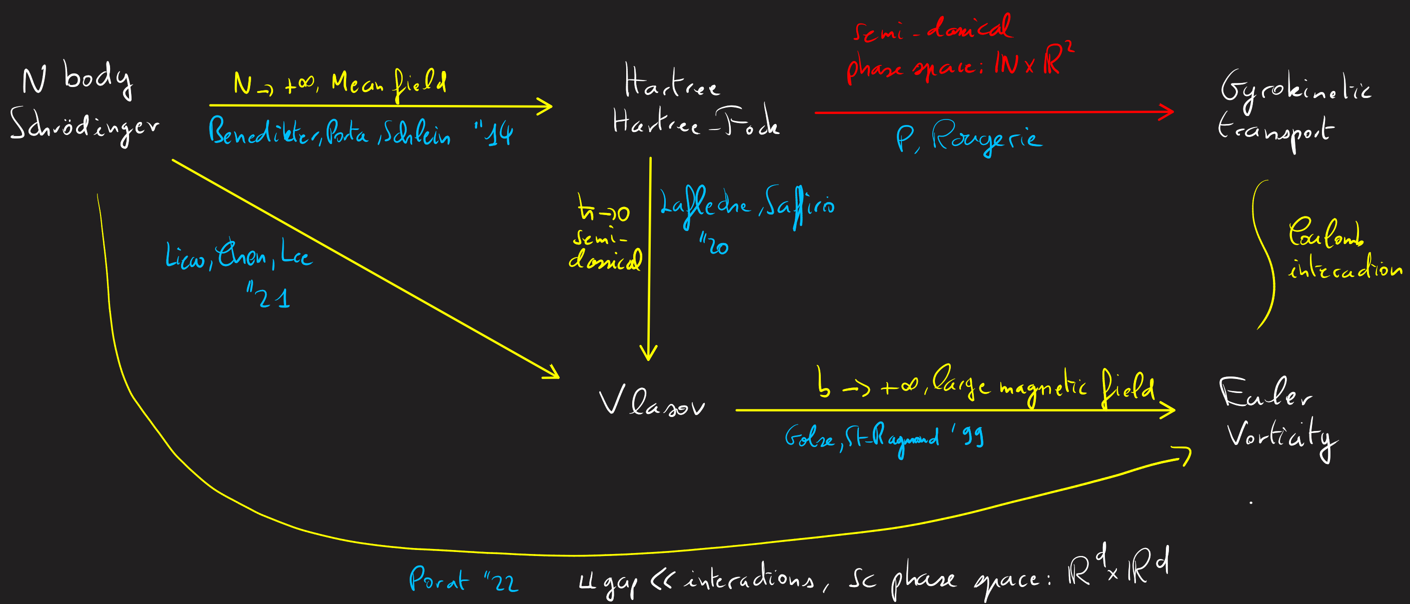
**Theorem: Convergence of densities**

Let $\gamma_b$ solves (H), assume $\gamma_b(0)$ is a FDM, $\mathrm{Tr}\left[\gamma_b(0)\left(\mathcal{L}_b + V + \frac{1}{2}\omega*\rho_{\gamma_b(0)}\right)\right]\leq C$, $V,\omega\in W^{4,\infty}(\mathbb{R}^2)$

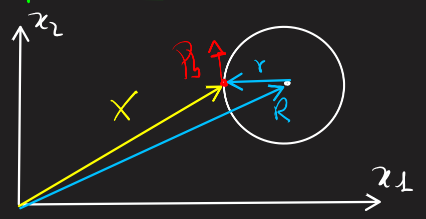
then, up to a subsequence $\begin{cases}\rho_{\gamma_b}\xrightarrow[b\to+\infty]{*}\rho\in\mathcal{M}(\mathbb{R}_+\times\mathbb{R}^2)\\ \rho_{\gamma_b}\xrightarrow[b\to+\infty]{*}\rho_0\in\mathcal{M}(\mathbb{R}^2)\end{cases}$ and $\forall\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^2)$,

$$\int_{\mathbb{R}_+\times\mathbb{R}^2}\rho\left(\partial_t\varphi + \nabla^\perp(V+\omega*\rho)\cdot\nabla\varphi\right) - \int_{\mathbb{R}^2}\varphi(0)\rho_0\,dz = 0$$

## Effective theories graph:



quantization:




| operators: | position | anihilation | creation |
| --- | --- | --- | --- |
| cyclotron movement | $r := \frac{P_b^\perp}{b}$ | $a := \frac{r_2 - i r_1}{\sqrt{2}\,\hbar_b}$ | $a^+ := \frac{r_2 + i r_1}{\sqrt{2}\,\hbar_b}$ |
| orbit center | $R := X - r$ | $\hat{b} := \frac{R_1 - iR_2}{\sqrt{2}\,\hbar_b}$ | $\hat{b}^+ := \frac{R_1 + iR_2}{\sqrt{2}\,\hbar_b}$ |

prop: $\mathcal{L}_b$ diagonalization

$[a,a^+] = [\hat b, \hat b^+] = \mathrm{Id}$, $\underbrace{[a,\hat b] = [a,\hat b^+] = 0}_{}$ solves $a\varphi_{0,0} = 0, \hat b\,\varphi_{0,0} = 0$

$$\varphi_{n,m} := \frac{(a^+)^n (\hat b^+)^m}{\sqrt{n!\,m!}}\varphi_{0,0} \quad \text{with } \varphi_{0,0}(x) := \frac{1}{\sqrt{2\pi}\,\hbar_b} e^{-\frac{|x|^2}{4\hbar_b^2}} \text{ is a basis of } L^2(\mathbb{R}^2) \text{ and } \Pi_n = \sum_{m\in\mathbb{N}} |\varphi_{n,m}\rangle\langle\varphi_{n,m}|$$

coherent state: Let $z\in\mathbb{C}$, $\varphi_{n,z} := e^{\frac{\bar z \hat b^+ - z\hat b}{\sqrt{2}\,\hbar_b}}\varphi_{n,0} = e^{-\frac{|z|^2}{4\hbar_b^2}} \sum_{m\in\mathbb{N}} \frac{\left(\frac{\bar z}{\sqrt{2}\,\hbar_b}\right)^m}{\sqrt{m!}}\varphi_{n,m}$ then $\bar{R}\varphi_{n,z} = \bar z\,\varphi_{n,z}$

$$\Pi_z := \sum_{n\in\mathbb{N}} |\varphi_{n,z}\rangle\langle\varphi_{n,z}| \text{ has kernel } \Pi_z(x,y) = \frac{1}{2\pi\hbar_b^2} e^{-\frac{|x-y|^2 - 2i(x^\perp\cdot y + 2z^\perp\cdot(x-y))}{4\hbar_b^2}}$$

So $\nabla_z^\perp \Pi_z(x,y) = \frac{i}{\hbar_b^2}(x-y)\Pi_z(x,y)$ or $\boxed{\nabla_z^\perp \Pi_z = \frac{1}{i\hbar_b^2}[\Pi_z, X]}$ $(*)$

Semi-classical limit:

To a density matrix $\gamma$ associate $m_\gamma(n,z) := \frac{1}{2\pi\hbar_b^2}\langle\varphi_{n,z}|\gamma\varphi_{n,z}\rangle$ (Hushimi measure)

Semi-classical density $\rho^x_{\gamma_b}(z) := \frac{1}{2\pi\hbar_b^2}\mathrm{Tr}[\gamma\Pi_z]$

Truncated semi-classical density $\rho^{x,\le N}_{\gamma_b}(z) := \sum_{n=0}^{N} m_\gamma(n,z)$

prop: convergence of $\rho_\gamma^{x,\le N}$

Let $\gamma_b$ be a FDM st. $\mathrm{Tr}[\gamma_b\mathcal{L}_b] \le C$, then $\forall\varphi\in C_c^\infty(\mathbb{R}^2)$,

$$\left|\int_{\mathbb{R}^2}\varphi(\rho_{\gamma_b} - \rho^{x,\le N}_{\gamma_b})\right| \le C(\varphi)\left(\frac{1}{\sqrt N}\underbrace{\mathrm{Tr}[\gamma_b\Pi_{\ge N}\mathcal{L}_b]^{1/2}}_{\to 0,\ N\to+\infty} + \underbrace{\sqrt N\,\hbar_b}_{\ll 1 \Leftrightarrow N \ll \frac{1}{\hbar_b}}\mathrm{Tr}[\gamma_b\mathcal{L}_b]^{1/2}\right)$$

prop: Gyrokinetic transport of $\rho^{x,\le N}_{\gamma_b}$

Let $t\in\mathbb{R}_+$, $\gamma_b(t)$ be a FDM, $W\in W^{4,\infty}(\mathbb{R}^2)$ assume

$\partial_t\gamma_b(t) = \frac{1}{i\hbar_b^2}\mathrm{Tr}[\mathcal{L}_b + W, \gamma_b(t)]$, $\mathrm{Tr}[\gamma_b(t)\mathcal{L}_b] \le C$

then there exists a choice of $1 \ll N \ll \frac{1}{\hbar_b^2}$ st.

$$\forall\varphi\in C_c^\infty(\mathbb{R}^2),\ \int_{\mathbb{R}^2}\varphi\left(\partial_t\rho^{x,\le N}_{\gamma_b(t)} + \nabla^\perp W\cdot\nabla\rho^{x,\le N}_{\gamma_b(t)}\right) \xrightarrow[b\to+\infty]{} 0$$

Central computation:

$$\partial_t\rho^x_{\gamma_b}(z) = \frac{1}{2\pi\hbar_b^2}\mathrm{Tr}[\Pi_z\partial_t\gamma_b] \overset{\text{Hartree}}{=} \frac{1}{2\pi\hbar_b^2}\cdot\frac{1}{i\hbar_b^2}\mathrm{Tr}[\Pi_z[\mathcal{L}_b + W, \gamma_b]] \overset{\text{Tr cyclicity}}{=} \frac{1}{2i\pi\hbar_b^4}\mathrm{Tr}[\gamma_b[\Pi_z, \mathcal{L}_b + W]]$$

$$= \frac{1}{2i\pi\hbar_b^4}\mathrm{Tr}[\gamma_b[\Pi_z, W]]$$

$$\nabla^\perp W(z)\cdot\nabla\rho^x_{\gamma_b}(z) = -\nabla W(z)\cdot\frac{1}{2\pi\hbar_b^2}\mathrm{Tr}[\gamma_b\nabla_z^\perp\Pi_z] \overset{(*)}{=} -\frac{1}{2i\pi\hbar_b^4}\nabla W(z)\cdot\mathrm{Tr}[\gamma_b[\Pi_z, X]]$$

So

$$\partial_t\rho^x_{\gamma_b}(z) + \nabla^\perp W(z)\cdot\nabla\rho^{x,\le N}_{\gamma_b}(z) = \frac{1}{2i\pi\hbar_b^4}\mathrm{Tr}\left[\gamma_b\left[\underbrace{\Pi_z, W - \nabla W(z)\cdot X}_{\Pi_z(x,y)(W(y) - W(x) - \nabla W(z)\cdot(y-x))}\right]\right]$$

Bonus:

$$\mathrm{Tr}\big[\gamma_b\,[\overbrace{\Pi_{n,z}}^{|\psi_{n,t}\rangle\langle\psi_{n,t}|},\underbrace{W-\nabla W(t)\cdot x}_{U_z}]\big]=\mathrm{Tr}[\gamma_b\Pi_{n,z}[\Pi_z,U]]+\mathrm{Tr}[\Pi_{n,t}\gamma_b[\Pi_z,U_z]]$$

$$|\mathrm{Tr}[\gamma_b[\Pi_z,U_z]]|\leqslant 2\sum_{n\in\mathbb{N}}{}'\,\mathrm{Tr}[\gamma_b^2\Pi_{n,z}]^{1/2}\,\|[\Pi_{n,t},U_z]\|_{\mathfrak{L}^2}$$

$$\leqslant\sum_{n\in\mathbb{N}}{}'\,\varepsilon_n\,\mathrm{Tr}[\gamma_b^2\Pi_{n,t}]+\sum_{n\in\mathbb{N}}{}'\frac{1}{\varepsilon_n}\int|\Pi_{n,t}(x,y)|^2\,(U_z(x)-U_z(y))^2\,dx\,dy$$

$$\leqslant\varepsilon\beta_b^2\sum_{n\in\mathbb{N}}{}'\,\mathrm{Tr}[\gamma_b\mathcal{L}_b\Pi_{n,t}]+\frac{1}{\varepsilon}\sum_{n\in\mathbb{N}}\frac{1}{E_n}\int|\Pi_{n,t}(x,y)|^2\,\underbrace{|U_z(x)-U_z(y)|^2}_{\leqslant C|x-y|^3\sim(\sqrt{n}\,\beta_b)^3}\,dx\,dy$$

$\varepsilon_n=\varepsilon E_n$

dc $W$ to 2nd order + IPP

$$\left|\int|\cdot|\,\varphi(z)\,dz\right|\leqslant\varepsilon\beta_b^4\,\mathrm{Tr}[\gamma_b\mathcal{L}_b]+\frac{1}{\varepsilon}\sum_{n\in\mathbb{N}}{}'\frac{n^?\beta_b^6}{E_n}$$

$\int\Pi_z=2\pi\beta_b^2$

$$\frac{1}{\beta_b^4}\left|\int\mathrm{Tr}[\gamma_b[\Pi_{\leqslant N,t},U_z]]\,\varphi(z)\,dz\right|\leqslant\varepsilon\,\mathrm{Tr}[\gamma_b\mathcal{L}_b]+\frac{\beta_b^2}{\varepsilon}\sum_{n=0}^{N}n=\varepsilon+\frac{N^2}{\varepsilon}\beta^2$$

$$=N\beta_b$$

$\varepsilon=N\beta_b$

- $\partial_t\rho+\nabla\cdot(u\rho)$, $\nabla^\perp\cdot u=\rho$, $\nabla\cdot u=0$

This implies $u=\nabla^\perp\varphi$, $\rho=\Delta\varphi$, $\varphi=\omega*\rho$ so $u=\nabla^\perp(\omega*\rho)$ if $\Delta\omega=\delta$

and $\partial_t\rho+\nabla\cdot(u\rho)=\partial_t\rho+u\cdot\nabla\rho=\partial_t\rho+\nabla^\perp(\omega*\rho)\cdot\nabla\rho$